Παράδειγμα Εαν $N_t \sim PP(\lambda)$ ειναι ο αριθμός των σεισμών παγκοσμίως στο $[0,t]$ κ' X_t ο αριθμός των σεισμών στο $[0,t]$ που υπερβαίνουν τα 5 Richter, δείξτε ότι $X_t \sim PP(\lambda p)$ όπου p = η πιθανότητα σεισμού ≥ 5.

$$P\{X_t=k\} = \sum_{n=0}^{\infty} P\{X_t=k \mid N_t=n\} P\{N_t=n\} =$$

$$= \sum_{n=k}^{\infty} P\{X_t=k \mid N_t=n\} P\{N_t=n\}$$

<u>όμως</u>: $[X_t \mid N_t=n] \sim Bin(n,p)$ έτσι

$$P\{X_t=k\} = \sum_{n=k}^{\infty} Bin(k \mid n,p) Po(n \mid \lambda t) =$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda t} (\lambda t)^n / n!$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \enclose{circle}{p^k} (1-p)^{n-k} \enclose{circle}{e^{-\lambda t}} (\lambda t)^n / \cancel{n!}$$

$$= \frac{e^{-\lambda t}(\lambda t p)^k}{k!} \sum_{n=k}^{\infty} \frac{[\lambda t(1-p)]^{n-k}}{(n-k)!} = \frac{e^{-\lambda t}(\lambda t p)^k}{k!} e^{\lambda t(1-p)}$$

$$= e^{-\lambda p t}(\lambda p t)^k / k! = P_o(k \mid \enclose{circle}{\lambda p} \cdot t).$$

<u>Παρατήρηση</u>: $T_i^X \overset{iid}{\sim} Exp(\lambda p)$

Παράδειγμα As θεωρήσουμε financial market όπου η άφιξη αγορών κάποιων χρεογράφων (risky assets) μοντελοποιείται απο $N_t \sim PP(\lambda)$. Το συνολικό αποτέλεσμα N_t οφείλεται στη συγκεντρωτική αγορά χρεογράφων από διαφορετικούς επενδυτές. Εαν υπάρχουν $\underline{n}$ ενεργοί επενδυτές $N_t = \sum_{j=1}^{n} N_t^j$ κ' ο j-επενδυτής αγοράζει με πιθανότητα p_j έτσι ώστε $\sum_{j=1}^{n} p_j = 1$. Δείξτε ότι $N_t^j \sim PP(\lambda p_j)$ $1 \leq j \leq n$ κ'

$N_t^j \perp N_t^i$, $i \neq j$

(i) $N_t^j \leq N_t, \forall t \geq 0 \Rightarrow N_0^j \leq N_0 = 0 \Rightarrow P\{N_0^j = 0\} = 1$

(ii)
$$P\{N_h^j = 1\} = \underbrace{P\{N_h = 1\}}_{\lambda h + o(h)} \underbrace{P\{N_h^j = 1 / N_h = 1\}}_{p_j} + \underbrace{P\{N_h \geq 2\}}_{o(h)} P\{N_h^j = 1 / N_h \geq 2\}$$

$$= \lambda p_j \cdot h + o(h) \qquad (2.1)$$

$$[N_h^j \geq 2 \Rightarrow N_h \geq 2] \Rightarrow \{N_h^j = 2\} \subseteq \{N_h = 2\} \Leftrightarrow$$

$$\Leftrightarrow P\{N_h^j \geq 2\} \leq P\{N_h \geq 2\} = o(h) \Rightarrow P\{N_h^j \geq 2\} = o(h)$$
$: (2.2)$

$$(2.1)(2.2) \Rightarrow P\{N_h^j = 0\} = 1 - \lambda p_j h + o(h)$$

(iii) $$P\{N_t^j - N_s^j = k\} = \sum_{n=k}^{\infty} P\{N_t^j - N_s^j = k \mid N_t - N_s = n\} P\{N_t - N_s = n\}$$

$$= \sum_{n=k}^{\infty} Bin(k \mid n, p_j) Po(n \mid \lambda(t-s)) = Po(n \mid \lambda p_j (t-s))$$

$$= P\{N_{t-s}^j = n\} \Rightarrow N_t^j - N_s^j \overset{d}{=} N_{t-s}^j .$$

(iv) $$P\{N_t^j - N_s^j = k \mid N_s^j - N_r^j = \ell\} = \sum_{n=k}^{\infty} P\{N_t^j - N_s^j = k \mid \cancel{N_s^j - N_r^j = \ell}, N_t - N_s = n\}$$

$$\times P\{N_t - N_s = n \mid \cancel{N_s^j - N_r^j = \ell}\}$$

$$= \sum_{n=k}^{\infty} Bin(k \mid n, p_j) Po(n \mid \lambda p_j (t-s) =$$

$$= Po(k \mid \lambda p_j (t-s)) = P\{N_t^j - N_s^j = k\}.$$

<u>Παράδειγμα</u> Έστω $N_t \sim PP(\lambda)$ κ' $N_t^j \sim PP(\lambda p_j)$ τ.ώ. $N_t = \sum_{j=1}^{n} N_t^j$ $N_t^i \perp\!\!\!\perp N_t^j$, $i \neq j$. Να βρεθεί ο αναμενόμενος χρόνος T παρουσίασης όλων των τύπων αφίξεων.

$$T_1^j = \inf\{t : N_t^j = 1\} = Y_1^j$$

$$T = \max_{1 \leq j \leq n} T_1^j$$

$$P\{T \leq t\} = P\{T_1^1 \leq t, \dots, T_1^n \leq t\} = \prod_{j=1}^{n} P\{T_1^j \leq t\}$$

$$= \prod_{j=1}^{n} (1 - e^{-\lambda p_j t}) \Rightarrow$$

$$\mathbb{E}[T] = \int_{\mathbb{R}^+} P\{T>t\}\,dt = \int_{\mathbb{R}^+} \left\{1-\prod_{j=1}^{n}(1-e^{-\lambda p_j \cdot t})\right\} dt$$

$$T = \max_{1\le j\le n} T_1^j = \sum_{i=1}^{N} T_i$$

$$\mathbb{E}[T] = \mathbb{E}\{\mathbb{E}[T|N]\} = \mathbb{E}\left\{\mathbb{E}\left[\sum_{i=1}^{N} T_i \,|\, N\right]\right\} =$$

$$= \mathbb{E}\{N\,\mathbb{E}[T_i]\} = \frac{1}{\lambda}\mathbb{E}[N] \Rightarrow \boxed{\mathbb{E}[N] = \lambda\,\mathbb{E}[T]}$$

<u>$n=2$, $\lambda=1$:</u> $\mathbb{E}[T] = \int_{\mathbb{R}^+}\{1-(1-e^{-p_1 t})(1-e^{-p_2 t})\}\,dt$

$$= 1 + \frac{1}{p(1-p)}, \quad p = p_1, \quad 1-p = p_2.$$

<u>Παράδειγμα</u>: Έστω ότι ταξιδιώτες φτάνουν στο σταθμο τραίνου συμφωνα με την $N_t \sim PP(\lambda)$. Εαν το τραίνο αναχωρεί σε χρόνο t, να υπολογιστεί ο αναμενόμενος χρόνος του χρόνου αναμονής όλων των ταξιδιωτών. $[(Y_1,\dots,Y_n)\,|\,N_t=n] \sim \mathcal{U}(0<u_1<\dots<u_n<t)$

$$\mathbb{E}\left[\sum_{i=1}^{N_t}(t-Y_i)\right] = \mathbb{E}\left\{\mathbb{E}\left[\sum_{i=1}^{N_t}(t-Y_i)\,\Big|\,N_t\right]\right\} \quad (4.1)$$

$$\mathbb{E}\left[\sum_{i=1}^{N_t}(t-Y_i)\,\Big|\,N_t=n\right] = nt - \mathbb{E}\left[\sum_{i=1}^{n} Y_i\,\Big|\,N_t=n\right] =$$

$$= nt - \mathbb{E}\left[\sum_{i=1}^{n} U_{(i)}\right] = nt - \mathbb{E}\left[\sum_{i=1}^{n} U_i\right] \underset{U_i \overset{iid}{\sim} U(0,t)}{=}$$

$$= nt - \frac{nt}{2} = \frac{nt}{2} \quad : (5.1)$$

$$(4.1)(5.1) \Rightarrow \mathbb{E}\left[\sum_{i=1}^{N_t} (t - Y_i)\right] = \mathbb{E}\left[\frac{t}{2} \cdot N_t\right] = \frac{\lambda t^2}{2}$$

<u>Παράδειγμα</u>: Έστω ότι μία άφιξη από $PP(\lambda)$ χαρακτηρίζεται σαν type-I με πιθανότητα $P(s)$ εάν συμβαίνει την χρονική στιγμή s κ' type-II με πιθανότητας $1 - P(s)$. Δείξτε ότι $N_t = N_t^{I} + N_t^{II}$ με $N_t^{I} \perp N_t^{II}$. $N_t^{I} \sim PP(\lambda p)$, $N_t^{II} \sim PP(\lambda(1-p))$ όπου $p = \frac{1}{t}\int_0^t P(s)ds$

$$P\{N_t^{I} = n, N_t^{II} = m\} = \sum_{k=0}^{\infty} P\{N_t^{I} = n, N_t^{II} = m \mid N_t = k\} P\{N_t = k\}$$

$$= P\{N_t^{I} = n, N_t^{II} = m \mid N_t = n+m\} P\{N_t = n+m\}$$

$$S = [T_1 \mid N_t = 1] \sim U(0,t) \Rightarrow$$

$$p = \mathbb{E}[P(S)] = \int_{\mathbb{R}} P(s) f_S(s) ds = \frac{1}{t}\int_0^t P(s) ds \Rightarrow$$

$$P\{\underset{\text{successes}}{N_t^I}=n,\ \underset{\text{failures}}{N_t^{II}}=m \mid N_t=m+n\} = Bin(n\mid m+n, p) =$$

$$= \binom{n+m}{n} p^n (1-p)^m$$

<u>Έτσι</u>

$$P\{N_t^I=n,\ N_t^{II}=m\} = \frac{(n+m)!}{n!\,m!} p^n (1-p)^m \cdot \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

$$= \frac{e^{-\lambda t p}(\lambda t p)^n}{n!} \cdot \frac{e^{-\lambda t(1-p)}[\lambda t(1-p)]^m}{m!}$$

$$= Po(n\mid \lambda p\cdot t)\, Po(m\mid \lambda(1-p)t) = P\{N_t^I=n\}\,P\{N_t^{II}=m\}$$

<u>Παράδειγμα</u>: Έστω ότι σύστημα υπόκειται σε διαταραχές σύμφωνα με την $N_t \sim PP(\lambda)$. Η i-οστή διαταραχή προκαλεί ζημιά $d_i \overset{iid}{\sim} d \sim f(\cdot)$. Η ζημιά όμως λόγω της διαταραχής μειώνεται εκθετικά με την πάροδο του χρόνου έτσι η d_i μετά από χρόνο t γίνεται $d_i e^{-\alpha t}$, $\alpha>0$. Εαν D_t είναι η συνολική ζημιά έως κ' χρόνο t τότε

$$D_t = \sum_{i=1}^{N_t} d_i e^{-\alpha(t-Y_i)}$$ όπου Y_i ο χρόνος αναμονής για την i στη σειρά Poisson διαταραχή.

$$\mathbb{E}[D_t \mid N_t=n] = \sum_{i=1}^{n} \mathbb{E}\{d_i e^{-\alpha(t-Y_i)} \mid N_t=n\} \overset{\text{ανεξ}}{=}$$

$$= \mathbb{E}[d]\, e^{-\alpha t}\, \mathbb{E}\Big\{\sum_{i=1}^{n} e^{\alpha Y_i} \mid N_t=n\Big\},\quad \underbrace{[Y_i \mid N_t=n]}_{\varphi_i} \underset{i\le n}{\sim} U(0,t)$$

$$\mathbb{E}\left[D_t \mid N_t = n\right] = e^{-\alpha t}\,\mathbb{E}[d]\,\underbrace{\mathbb{E}\left\{\sum_{i=1}^{n} e^{\alpha u_i}\right\}}_{\sum_{i=1}^{n}\frac{1}{t}\int_0^t e^{\alpha x}dx = \frac{n}{t\alpha}\left(e^{\alpha t}-1\right)} = \mathbb{E}[d]\cdot\frac{n}{t\alpha}\left(1-e^{-\alpha t}\right)$$

$$\mathbb{E}\left[D_t\right] = \mathbb{E}\left\{\frac{\mathbb{E}[d]}{t\alpha}N_t\left(1-e^{-\alpha t}\right)\right\} = \frac{\lambda\,\mathbb{E}[d]}{\alpha}\left(1-e^{-\alpha t}\right)$$

<u>Παραδειγμα</u> Εαν $\{W_t\}_{t\geq 0}$ η διαδικασια Weiner ορίζουμε $\{X_t\}_{t\in[0,1]}$ με $X_t = W_t - tW_1$. Να βρεθεί η κατανομη της X_t.

$$P\{W_t - tW_1 \in A\} = \mathbb{E}\left\{P\{W_t - tW_1 \in A \mid W_1\}\right\} =$$

$$= \int_{\mathbb{R}} P\{W_t - tW_1 \in A \mid W_1 = u\}\,N(u\mid 0,1)\,du =$$

$$= \int_{\mathbb{R}}\left\{\int_{A+tu} f_{W_t\mid W_1}(v\mid u)\,dv\right\}N(u\mid 0,1)\,du$$

γνωρίζουμε ότι: $\left[W_t \mid W_\tau = y\right] \underset{t<\tau}{\sim} N\left(\frac{t}{\tau}y,\ \frac{t}{\tau}(\tau-t)\right)$: (7.1)

έτσι $\int_{A+tu} f_{W_t\mid W_1}(v\mid u)\,dv = \int_{A+tu} N(v\mid tu,\ t(1-t))\,dv = \int_A N(v\mid 0,\ t(1-t))\,dv$

Έτσι $P\{W_t - tW_1 \in A\} = \overbrace{\int_{\mathbb{R}} N(u/0,1)\,dv}^{1} \int_A N(v/0,t(1-t))\,dv \Rightarrow$

$\Rightarrow X_t = W_t - tW_1 \sim N(0, t(1-t))$

Το covariance της $\{X_t\}_{t\in[0,1]}$ είναι: για $1 \le s \le t \le 1$

$$Cov(X_s, X_t) = Cov(W_s - sW_1, W_t - tW_1) =$$

$$= Cov(W_s, W_t) - t\,Cov(W_s, W_1) - s\,Cov(W_1, W_t) + st\,Cov(W_1, W_1)$$

$$= s\wedge t - t\cdot(s\wedge 1) - s(1\wedge t) + st(1\wedge 1) = s(1-t)$$

Δηλ. Η X_t είναι μια διαδικασία ορισμένη στο $T=[0,1]$ που είναι Gaussian με μέση τιμή 0 κ' covariance $s(1-t)$, όταν $1\le s\le t\le 1$.

Παρατηρούμε ότι: $P\{X_1 = 0\} = P\{X_0 = 0\} = 1$ κ' ότι

$$X_t = W_t - tW_1 \overset{d}{=} [W_t,\ t\in[0,1] \,|\, W_1 = 0]$$

Πρόγψηση: $Cov[W_s, W_t | W_1 = 0] =$

$$= \mathbb{E}[W_sW_t|W_1=0] - \underbrace{\mathbb{E}[W_s|W_1=0]}_{0}\cdot\underbrace{\mathbb{E}[W_t|W_1=0]}_{0} \Leftarrow (7\cdot 1)$$

$$= \mathbb{E}\{\mathbb{E}[W_sW_t|W_1=0, W_t]\,|\,W_1=0\} =$$

$$= \mathbb{E}\left\{ \underbrace{\mathbb{E}\left[W_s W_t \mid W_t\right]}_{W_t \underbrace{\mathbb{E}[W_s \mid W_t]}_{\frac{s}{t} W_t}} \mid W_1 = 0 \right\} = \frac{s}{t}\mathbb{E}\left[W_t^2 \mid W_1 = 0\right] \quad : (9.1)$$

$$(7.1) \Rightarrow W_t \mid W_1 = 0 \underset{t<1}{\sim} N(0, t(1-t))$$

$$(9.1) \Rightarrow \mathrm{Cov}\left[W_s, W_t \mid W_1 = 0\right] = s(1-t).$$

Παράδειγμα: (i) Δείξτε ότι

$$\mathrm{Cov}(X,Y) = \mathbb{E}\{\mathrm{Cov}(X,Y|Z)\} + \mathrm{Cov}\left[\mathbb{E}(X|Z), \mathbb{E}(Y|Z)\right]$$

(ii) Εάν $Z_t = \sum_{i=1}^{N_t} X_i$, $X_i \overset{iid}{\sim} X \sim f_X(\cdot)$

βρείτε το covariance της Z_t.

(i) $\mathrm{Cov}(X,Y|Z) = \mathbb{E}[XY|Z] - \mathbb{E}[X|Z]\mathbb{E}[Y|Z] \Rightarrow$

$\Rightarrow \mathbb{E}\{\mathrm{Cov}(X,Y|Z)\} = \mathbb{E}[XY] - \mathbb{E}\{\mathbb{E}[X|Z]\mathbb{E}[Y|Z]\}$

$\mathrm{Cov}\left[\mathbb{E}(X|Z), \mathbb{E}(Y|Z)\right] = \mathbb{E}\{\mathbb{E}(X|Z)\mathbb{E}(Y|Z)\} - \mathbb{E}\{\mathbb{E}(X|Z)\}\mathbb{E}\{\mathbb{E}(Y|Z)\}$

$= \mathbb{E}\{\mathbb{E}(X|Z)\mathbb{E}(Y|Z)\} - \mathbb{E}[X]\mathbb{E}[Y]$

$:(9.1)$

(ii) $\mathrm{Cov}(Z_s, Z_t) \overset{s<t}{=} \mathbb{E}\{\mathrm{Cov}(Z_s, Z_t | N_t)\} + \mathrm{Cov}\{\mathbb{E}[Z_s|N_t], \mathbb{E}[Z_t|N_t]\}$

$\mathrm{Cov}(Z_s, Z_t | N_t) = \mathbb{E}\{Z_s Z_t | N_t\} - \mathbb{E}\{Z_s|N_t\}\mathbb{E}\{Z_t|N_t\} \quad : (9.2)$

$$\mathbb{E}\left[Z_s Z_t \,|\, N_t=n\right] = \mathbb{E}\left\{ \sum_{i=1}^{N_s} X_i \cdot \sum_{j=1}^{N_t} X_j \,|\, N_t = n \right\} = \mathbb{E}\left\{ \sum_{i=1}^{N} X_i \sum_{j=1}^{n} X_j \,|\, N_t = n \right\}$$

οπου $N = [N_s | N_t = n] \sim Bin(n, s/t)$, $s<t$. (10.1)

Απο την (10.1) γίνεται εμφανες οτι κ' πάλι θα πρέπει να χρησιμοποιήσουμε επαναλαμβανόμενη (υπο-συνθήκη) μέση τιμή δηλαδή:

$$\mathbb{E}\left[Z_s Z_t \,|\, N_t=n\right] = \mathbb{E}\left\{ \mathbb{E}\left[Z_s Z_t \,|\, N, N_t = n\right] \,|\, N_t = n \right\} : (10.2)$$

$$\mathbb{E}\left[Z_s Z_t \,|\, N = m, N_t = n\right] = \mathbb{E}\left\{ \sum_{i=1}^{m} X_i \sum_{j=1}^{n} X_j \right\} =$$

$$= \mathbb{E}\left\{ \sum_{i=1}^{m} X_i \cdot \left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \right) \right\} = \mathbb{E}\left\{ \left(\sum_{i=1}^{m} X_i \right)^2 + \sum_{i=1}^{m} \sum_{j=m+1}^{n} X_i X_j \right\}$$

$$= \mathbb{E}\left\{ \sum_{i=1}^{m} X_i^2 + 2 \sum_{1 \le i < j \le m} X_i X_j + \sum_{i=1}^{m} \sum_{j=m+1}^{n} X_i X_j \right\}$$

$$= m\,\mathbb{E}[X^2] + 2 \cdot \binom{m}{2} \mathbb{E}[X]^2 + m(n-m)\,\mathbb{E}[X]^2$$

$$= m\,\mathbb{E}[X^2] + m(n-1)\,\mathbb{E}[X]^2$$

$$(10.2) \Rightarrow \mathbb{E}\left[Z_s Z_t \,|\, N_t = n\right] = n\frac{s}{t}\mathbb{E}[X^2] + n\frac{s}{t}(n-1)\,\mathbb{E}[X]^2 \Rightarrow$$

$$\Rightarrow \mathbb{E}\left[Z_s Z_t\right] = \mathbb{E}\left\{ \frac{s}{t} N_t \mathbb{E}[X^2] + \frac{s}{t} N_t (N_t - 1) \mathbb{E}[X]^2 \right\} =$$

$$= \frac{s}{t}\mathbb{E}[N_t]\,\mathbb{E}[X^2] + \frac{s}{t}\left\{ \mathbb{E}[N_t^2] - \mathbb{E}[N_t] \right\} \mathbb{E}[X]^2$$

$$= \lambda s\,\mathbb{E}[X^2] + \lambda^2 st\,\mathbb{E}[X]^2 : (10.3)$$

$$\mathbb{E}[Z_s \mid N_t=n] = \mathbb{E}\{\mathbb{E}[Z_s \mid N, N_t=n] \mid N_t=n\} =$$

$$= \mathbb{E}\{N\,\mathbb{E}[X] \mid N_t=n\} = n \cdot \frac{s}{t}\,\mathbb{E}[X] \Rightarrow$$

$$\Rightarrow \mathbb{E}[Z_s \mid N_t] = \frac{s}{t} \cdot N_t\,\mathbb{E}[X] \quad , \quad \mathbb{E}[Z_t \mid N_t] = N_t\,\mathbb{E}[X]$$

έτσι

$$\mathrm{Cov}\{\mathbb{E}(Z_s \mid N_t), \mathbb{E}(Z_t \mid N_t)\} = \mathrm{Cov}\left\{\frac{s}{t} N_t\,\mathbb{E}[X], N_t\,\mathbb{E}[X]\right\}$$

$$= \frac{s}{t}\,\mathbb{E}[X]^2\,\mathrm{Cov}(N_t, N_t) = \frac{s}{t}\,\mathbb{E}[X]^2\,\mathrm{Var}[N_t] = \lambda s\,\mathbb{E}[X]^2 \qquad (11{\cdot}1)$$

$$\mathbb{E}\{\mathrm{Cov}(Z_s, Z_t \mid N_t)\} = \mathbb{E}\left\{\frac{s}{t} N_t\,\mathbb{E}[X^2] + \frac{s}{t} N_t(\cancel{N_t}-1)\,\mathbb{E}[X]^2 - \right.$$

$$\left. - \frac{s}{t} \cancel{N_t}\,\mathbb{E}[X] \cdot N_t\,\mathbb{E}[X]\right\} = \mathbb{E}\left\{\frac{s}{t} N_t\,\mathrm{Var}[X]\right\} = \lambda s\,\mathrm{Var}[X] \qquad (11.2)$$

$$(9{\cdot}1)(11{\cdot}1)(11{\cdot}2) \Rightarrow \mathrm{Cov}(Z_s, Z_t) \overset{s<t}{=} \lambda s\,\mathrm{Var}[X] + \lambda s\,\mathbb{E}[X]^2 = \lambda s\,\mathbb{E}[X^2]$$

Απο συμμετρία έχουμε $\mathrm{Cov}(Z_s, Z_t) \overset{s>t}{=} \lambda t\,\mathbb{E}[X^2] \Rightarrow$

$$\Rightarrow \forall\ s,t>0 : \mathrm{Cov}(Z_s, Z_t) = \lambda \cdot s \wedge t \cdot \mathbb{E}[X^2]$$

Παράδειγμα Δίνεται η σδ $\{K_t\}_{t\in[0,m]}$ κ'

$K_t = \inf\{n : \sum_{i=1}^{n} u_i > t\}$ όπου $u_i \overset{iid}{\sim} U(0,m)$. Να βρεθεί $\mu(t) = \mathbb{E}[K_t]$.

$$\mu(t) = \mathbb{E}[K_t] = \mathbb{E}\{\mathbb{E}[K_t|u_1]\} = \int_{\mathbb{R}} \mathbb{E}[K_t|u_1=y]\,U(y|0,m)\,dy$$

$$= \frac{1}{m}\int_0^m \mathbb{E}[K_t|u_1=y]\,dy$$

$$\left.\begin{array}{l} y>t \Rightarrow \mathbb{E}[K_t|u_1=y]=1 \\ y\le t \Rightarrow \mathbb{E}[K_t|u_1=y] = 1+\mathbb{E}[K_{t-y}] \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mu(t) = \frac{1}{m}\int_0^m \{1(y>t) + [1+\mu(t-y)]\,1(y\le t)\}\,dy =$$

$$= \frac{1}{m}\left\{m-t+t+\int_0^t \mu(t-y)\,dy\right\}$$

$$\Leftrightarrow \mu(t) = 1+\frac{1}{m}\int_0^t \mu(t-y)\,dy \Rightarrow \mu'(t) = \frac{1}{m}\left[\mu(0)+\int_0^t \mu'(t-y)\,dy\right]$$

$$\underset{z=t-y}{\Leftrightarrow} \mu'(t) = \frac{1}{m}\left[\mu(0)+\int_0^t \mu'(z)\,dz\right]$$

$$\Leftrightarrow \mu'(t) = \mu(t)/m \Rightarrow \int_{u=0}^{t}\frac{d\mu(u)}{\mu(u)} = \int_{u=0}^{t}\frac{du}{m} \Leftrightarrow$$

$$\Leftrightarrow \mu(t) = \mu(0)\,e^{t/m}$$ αλλά $\mu(0) = \mathbb{E}[K_0] = \mathbb{E}[1] = 1 \Rightarrow$

$$\Rightarrow \mu(t) = e^{t/m}$$

□

Παρατήρηση: (i) $$\frac{d}{dt}\int_{\alpha(t)}^{b(t)} f(t,y)\,dy = b'(t)f(t,b(t)) - \alpha'(t)f(t,\alpha(t)) + \int_{\alpha(t)}^{b(t)}\frac{\partial}{\partial t}f(t,y)\,dy$$

(ii) Ισχύει πάντα ότι εάν

συνεχής τμ.

$$K_t^x = \inf\left\{n : \sum_{i=1}^{n} x_i > t\right\} \qquad x_i \overset{iid}{\sim} f(\cdot), \quad x_i > 0$$

$$N_t^x = \sup\left\{n : \sum_{i=1}^{n} x_i \le t\right\}$$

Τότε $P\{K_t^x - N_t^x = 1\} = 1$

Άσκηση Στις σημειώσεις SP1_06.pdf στην άσκηση της σελ. $\boxed{2}$ δείξαμε ότι $Cov(N_s, N_t) = \lambda \cdot s \wedge t$ χρησιμοποιώντας ότι $[N_s | N_t = n] \overset{s<t}{\sim} Bin(n, s/t)$. :(13.1)
Δείξτε ότι $Cov(N_s, N_t) = \lambda \cdot s \wedge t$ μη κάνοντας χρήση της (13.1).

Έστω $s<t$ τότε $E[N_s N_t] = E\{E[N_s N_t | N_s]\} =$

$= E[N_s E[N_t | N_s]]$:(13.2)

$$P\{N_t = n | N_s = m\} = P\{N_t - N_s = n-m | N_s = m\} = P\{N_t - N_s = n-m\}$$

$$= P\{N_{t-s} = n-m\} = P_o(n-m | \lambda(t-s))$$

$$E[N_t | N_s = m] = \sum_{n=0}^{\infty} n \cdot P\{N_t = n | N_s = m\} = \sum_{n=m}^{\infty} n P\{N_t = n | N_s = m\}$$

$$= \sum_{n=m}^{\infty} n \cdot P_o(n-m | \lambda(t-s)) = \sum_{n=m}^{\infty} n \cdot e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^{n-m}}{(n-m)!}$$

$$\overset{k=n-m}{=\!=} e^{-\lambda(t-s)} \sum_{k=0}^{\infty} (k+m) [\lambda(t-s)]^k / k! =$$

$$= e^{-\lambda(t-s)} \left\{ \sum_{k=0}^{\infty} k \cdot [\lambda(t-s)^k] / k! + m \sum_{k=0}^{\infty} [\lambda(t-s)]^k / k! \right\}$$

$$= e^{-\lambda(t-s)}\left\{\sum_{k=1}^{\infty}\frac{k}{k!}[\lambda(t-s)]^{k} + m\, e^{\lambda(t-s)}\right\}$$

$$= e^{-\lambda(t-s)}\left\{\lambda(t-s)\underbrace{\sum_{k=1}^{\infty}\frac{[\lambda(t-s)]^{k-1}}{(k-1)!}}_{e^{\lambda(t-s)}} + m\, e^{\lambda(t-s)}\right\} = \lambda(t-s)+m$$

$$\Rightarrow \boxed{\mathbb{E}[N_t \mid N_s=m] = \lambda(t-s)+m} \Rightarrow \mathbb{E}[N_t \mid N_s] \overset{s<t}{=} \lambda(t-s)+N_s$$

:(14·1)

$$(13\cdot 2) \Rightarrow \mathbb{E}[N_s N_t] = \mathbb{E}[N_s(\lambda(t-s)+N_s)] =$$

$$= \lambda(t-s)\mathbb{E}[N_s] + \mathbb{E}[N_s^2] = \lambda(t-s)\cdot\lambda s + (\lambda s+\lambda^2 s^2)$$

$$= \lambda s + \lambda^2 s t$$

<u>Έτσι</u>: $\mathrm{Cov}(N_s,N_t) \overset{s<t}{=} (\lambda s+\lambda^2 st) - (\lambda s)(\lambda t) = \lambda s$

<u>απο συμμετρία</u>: $\mathrm{Cov}(N_s,N_t) = \lambda\cdot s\wedge t \;,\; \forall\, s,t\geq 0$

□

<u>Παρατήρηση</u>: Απο την (14·1) $\Rightarrow \mathbb{E}[N_t-\lambda t \mid N_s] = N_s-\lambda s$

<u>Άσκηση</u> Εστω $X_t \sim PP(\lambda)$, $Y_t \sim PP(s)$ κ' $X_t \perp Y_t$ τότε $Z_t = X_t+Y_t \sim PP(\lambda+s)$

$$P\{Z_t=n\} = P\left(\bigcup_{k=0}^{n}\{X_t=k, Y_t=n-k\}\right) = \sum_{k=0}^{n} P\{X_t=k, Y_t=n-k\}$$

$$= \sum_{k=0}^{n} P\{X_t=k\}P\{Y_t=n-k\} = \sum_{k=0}^{n} P_o(k\mid\lambda t)\, P_o(n-k\mid st)$$

$$= \sum_{k=0}^{n} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\cdot e^{-st}\frac{(st)^{n-k}}{(n-k)!} = \frac{e^{-(\lambda+s)t}}{n!}(st)^n \sum_{k=0}^{n}\binom{n}{k}(\lambda/s)^k$$

$$= \frac{e^{-(\lambda+s)t}}{n!} s^n t^n \left(1+\frac{\lambda}{s}\right)^n = e^{-(\lambda+s)t}\{(\lambda+s)t\}^n/n!$$

$$= P_o(n|\lambda+s)$$

□

Ασκηση: Δο΄ (i) Εαν $X_t \sim PP(\lambda)$ κ΄ $Y_t \sim PP(s)$ με $X_t \perp\!\!\!\perp Y_t$ τότε $Z_t = X_t + Y_t$ είναι $Z_t \sim PP(\lambda+s)$

(ii) Εάν $X_t \sim CPP(\lambda, f(\cdot))$ κ΄ $Y_t \sim CPP(\mu, g(\cdot))$ με $X_t \perp\!\!\!\perp Y_t$ τότε $Z_t = X_t + Y_t$ είναι $Z_t \sim CPP(\Lambda, \frac{\lambda}{\Lambda}f(\cdot) + \frac{\mu}{\lambda}g(\cdot))$ όπου $\Lambda = \lambda+\mu$.

(iii) Εαν $Y_t \sim CPP(\lambda, Exp(\mu))$ να βρεθεί η $M_{Y_t}(s)$ κ΄ η $\mathbb{E}[Y_t]$

(i) Προηγούμενη άσκηση

(ii) Παραδ. σελ $\boxed{25}$: SP1_05.pdf.

(iii) $Y_t = \sum_{i=1}^{N_t} X_i$; $X_i \overset{iid}{\sim} Exp(\mu) \Leftrightarrow Y_t \sim CPP(\lambda, Exp(\mu))$

$$M_{Y_t}(s) = \mathbb{E}\left\{\mathbb{E}\left[e^{s\sum_{i=1}^{N_t} X_i} \,\middle|\, N_t\right]\right\} = \mathbb{E}\left\{\mathbb{E}\left[e^{sX_1} \cdots e^{sX_{N_t}} \,\middle|\, N_t\right]\right\}$$

$$\Rightarrow M_{Y_t}(s) = \mathbb{E}\left\{ M_X(s)^{N_t} \right\}$$ όπου $X \sim Exp(\mu)$

$$\left.\begin{aligned}
&\Leftrightarrow M_{Y_t}(s) = \mathbb{E}\left\{ \exp\left\{ N_t \log\left(M_X(s)\right)\right\}\right\} = \\
&M_X(s) = \int_{\mathbb{R}^+} e^{sx}\left(\mu e^{-\mu x} dx\right) = \frac{\mu}{\mu - s}, \quad s < \mu
\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}
&M_{Y_t}(s) = M_{N_t}\left(\log\left(\frac{\mu}{\mu-s}\right)\right) \\
&M_{N_t}(s) = \sum_{K=0}^{\infty} e^{Ks} \cdot \left\{ e^{-\lambda t} \frac{(\lambda t)^K}{K!} \right\} = e^{-\lambda t} \sum_{K=0}^{\infty} \frac{(\lambda t e^{s})^K}{K!} \\
&\quad = e^{-\lambda t} e^{\lambda t e^{s}} = e^{\lambda t (e^{s} - 1)}
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \boxed{M_{Y_t}(s) = \exp\left\{ \frac{\lambda t s}{\mu - s} \right\}, \quad s < \mu}$$

$$M'_{Y_t}(0) = \mathbb{E}[Y_t] = \left\{ \frac{\lambda t s}{\mu - s} \right\}' \exp\left\{ \frac{\lambda t s}{\mu - s} \right\}\Bigg|_{s=0} =$$

$$= \frac{\lambda t \mu}{(\mu - s)^2} M_{Y_t}(s)\Bigg|_{s=0} = \frac{\lambda t}{\mu} M_{Y_t}(0) = \frac{\lambda t}{\mu} \Rightarrow$$

$$\Rightarrow \boxed{\mathbb{E}[Y_t] = \lambda t / \mu}$$

□